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Replica field theory for deterministic models: II. A non-random spin glass with glassy behaviour

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Abstract. We introduce and study a model which admits a complex landscape without containing quenched disorder. Continuing our previous investigation we introduce a disordered model which allows us to reconstruct all the main features of the original phase diagram, including a low-T spin-glass phase and a complex dynamical behaviour.

1. Introduction

In our companion paper [1] (which in the following we will denote as I) we have started (at the same time as Bouchaud and Mezard in [2]) a study of the role of replica field theory when applied to the study of systems which *do not* contain quenched disorder (for further connected work which helps clarify this issue see [3,4].

The immediate starting point which prompted our investigation (I) was a model of binary sequences with low autocorrelation, as originally discussed by Golay and Bernasconi [5,6]. The model was for us a prototype of a system which does not contain quenched random disorder, but has an interesting spin-glass-like low-T structure (for general discussions about disordered systems see [7–9]). We have shown that replica theory allows us to gather information about the full phase diagram of the theory, excluding only the very low-T behaviour, which is determined by various factors, including the cardinality of the number of spins of the system, N. Indeed we have shown in I that replica theory can allow a study of the full deterministic model, and does not have to be limited to an approximate form.

Apart from such a direct application, we have discussed in I a more general valence of such an approach. The ability of investigating deterministic systems with a complex landscape is an important bonus. We also stress that we are still lacking a comprehensive description of the glass state, and that such an approach seems a good candidate for this task.

In the following we will discuss a new class of models without quenched disorder. They derive quite directly from the ones studied in I, by noticing the peculiar role the Fourier transform is playing (we will discuss this point in some detail in section 2). We will find that these models behave in a way which appears to be relevant to the description of the glass state.

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We will define the first model (the *sine model*) by the Hamiltonian

$$H_S \equiv \sum_{x=1}^{N} \left\{ \sum_{y=1}^{N} [S_x(\sigma_y)] - \sigma_x \right\}^2 \tag{1}$$

where

$$S_x(\sigma_y) \equiv \frac{1}{\sqrt{N}} \sin\left(\frac{2\pi xy}{N}\right)\sigma_y \tag{2}$$

and the spin variables σ_x take the values ± 1. We define the analogous *cosine model* by the Hamiltonian H_C

$$H_C \equiv \sum_{x=1}^{N} \left\{ \sum_{y=1}^{N} [C_x(\sigma_y)] - \sigma_x \right\}^2 \tag{3}$$

where

$$C_x(\sigma_y) \equiv \frac{1}{\sqrt{N}} \cos\left(\frac{2\pi xy}{N}\right)\sigma_y . \tag{4}$$

Let us anticipate a discussion of the phase diagram of the model. We will see that a very important role is played by the case where $(2N+1)$ is prime (and N is odd for the sine model and even for the cosine model). In this case the thermodynamical limit of the partition function is anomalous. Indeed, we will show that from the thermodynamical point of view for prime values of $(2N+1)$ our models undergo a first-order transition at a temperature T_C. We find such a *crystallization* transition only in the case of prime $(2N+1)$. At T_C the system goes from a disordered state to a highly ordered one. The specific heat in the low-temperature crystalline state is extremely small.

The system, however, has a metastable phase whose internal energy is regular at T_C. When we start from high T with local Monte Carlo dynamics, and we decrease T with some kind of annealing procedure, we pass through T_C without any noticeable change in the thermodynamical quantities.

At a lower temperature T_G, within the metastable phase, there is a transition to a *glassy* phase (a second-order phase transition). This transition exists for generic values of N. In the glassy phase the system may exist in many different equilibrium metastable states. Here there are many states which survive with finite probability in the infinite-volume limit (in other words replica symmetry is broken). In this phase the system freezes and thermodynamic fluctuations (for instance of the energy and of the magnetization) are very small. The behaviour of the system at the glass transition can be understood in the framework of replica theory. It is remarkable that the glass transition temperature T_G is the temperature where the entropy in the metastable phase becomes nearly equal to the entropy in the glassy phase (i.e. very close to zero).

We stress again that the crystalline phase only exists for $(2N+1)$ prime, N odd for the sine model and even for the cosine model. On the other hand the behaviour of the system in the high-temperature phase and in the metastable phase is generic, and does not depend on the cardinality of $(2N+1)$.

In section 2 we will briefly describe the genesis of this model, after our paper I. We will also discuss the low-T phase of the low autocorrelation model, mainly by using number theory. We will again be quite sketchy, inviting the interested reader to consult I for a more detailed discussion. In section 3 we will define a model containing quenched disorder, which we will eventually dissect by replica theory, and show to give a fair description of many features of our deterministic models. We will eventually show that the random

model and the deterministic one do basically coincide, apart for minor details such as the non-generic existence of the crystalline phase in the deterministic models.

In section 4 we describe our replica computation. In section 5 we analyse the saddle-point equations. We describe the replica-symmetric and the one-step replica-broken solution. In section 6 we discuss the so-called marginality condition. In section 7 we illustrate our numerical simulations of the models with quenched random disorder, and in section 8 the numerical simulations of the deterministic models. In section 9 we discuss the mean-field equations for the deterministic models. In section 10 we draw our conclusions. In the final appendix we present the technical details of a computation concerning the marginal stability.

2. The genesis of our models

In order to introduce the models we have defined in the previous section, and which we will study in the following, let us recall some basic definitions from I, and repeat briefly the reasoning which leads to the exhibition of the exact ground state of the model for some particular values of the number of spins. The reader in need of further details should consult I and [10].

The low autocorrelation model is based on sequences of length p of spin variables $\sigma_x = \pm 1$, with $x = 1, p$, and on the Hamiltonian

$$H = \frac{1}{p-1} \sum_{k=1}^{p-1} C_k^2 \tag{5}$$

where the C_k are the correlations at distance k, defined as

$$C_k \equiv \sum_{j=1}^{p} \sigma_j \sigma_{j+k} \tag{6}$$

where we are taking periodic boundary conditions (this is, in the terminology of I, the *periodic* model), i.e. the indices are always summed modulo p. In this way the indices which address the σ variables always belong, as they should, to the interval $[1, p]$. It is useful to rewrite the Hamiltonian as

$$H = \frac{1}{p-1} \sum_{k=1}^{p} \left(|B(k)|^4 - 1 \right) + 1 \tag{7}$$

where the Fourier transform is defined as

$$B(k) \equiv \frac{1}{\sqrt{p}} \sum_{x=1}^{p} e^{i(2\pi k/p)x} \sigma_j \tag{8}$$

and i is the imaginary unit. The thermodynamics of the model can be reconstructed thanks to the partition function at inverse temperature $\beta \equiv 1/T$ in the volume p

$$Z_p(\beta) \equiv \sum_{\{\sigma\}} e^{-\beta H\{\sigma\}}. \tag{9}$$

An interesting way to look at the Hamiltonian (5) is to consider it as a particular form of a fully frustrated 4-spin interaction. Here only the 4-spin terms, which are contained in a square of two-point correlation functions, appear. This point of view has been useful in I to show that replica theory can be a reasonable tool for investigating deterministic models.

It is remarkable that for prime values of p, such that $p = 4n+3$, it is possible to exhibit in an explicit way one ground state of the system. Let us construct such a ground-state configuration. Following Legendre [10] we set $\sigma_p = 0$ and

$$\sigma_j = j^{\frac{1}{2}(p-1)} \bmod p\,. \tag{10}$$

In this way σ_j is $+1$ or -1, if $j < p$. Indeed, a theorem by Fermat [10] tells us that if j is not a multiple of p, $j^{(p-1)} = 1$, mod(p) and therefore $j^{\frac{1}{2}(p-1)} = \pm 1$.

We will evaluate the energy of this sequence and only at the end will we impose that $\sigma_p = \pm 1$ on the last site p. It is well known that for this sequence all the correlations C_k are equal to -1 [10]. It is also remarkable (and the crux of this paper) that on such a sequence the Fourier transform is given by

$$B(k) = G(p)\,\sigma_k \tag{11}$$

where, according to Gauss [10], $G(p) = 1$ for $p = 4n+1$ and $G(p) = -\mathrm{i}$, for $p = 4n+3$. This Gauss theorem makes it easy to verify that the configurations we have exhibited have energy 1 (the lowest possible energy for odd values of p). Now we change the last spin to ± 1. It is easy to verify that after doing that the energy of configurations with p of the form $4n+3$ stays unchanged at 1, while for $p = 4n+1$ the energy grows to 5. It is clear now that for p prime of the form $4n+3$ we have exhibited a true ground state of the low autocorrelation model.

By using Gauss' theorem for Fourier transforms of Legendre sequences we are now able to define a simple model with 2-spin interaction which has the same ground state as the 4-spin interaction low autocorrelation model. We are ignoring here the presence of the spin with value zero. The new Hamiltonian has the form

$$H = \sum_x |G(p)\,\sigma_x - B(x)|^2\,. \tag{12}$$

We can further simplify the model by noticing that the sequence of the σ in the ground state is symmetric or antisymmetric around the point $\frac{1}{2}(p-1)$, depending on the value of $G(p)$. This allows us to define two new models with half the number of degrees of freedom which continue to admit (for selected p values) the ground state we have written; two such models are the sine and the cosine model we have defined in our introduction.

Hopefully we have to some extent clarified the nature of our two models. Now we can proceed to study them.

3. The disordered model

At this point it is natural to introduce a model which contains quenched disorder. Our companion paper (I) justifies this approach, in detail. By studying a suitable disordered model we try to understand how general a very specific 2-spin interaction is, like, for example, the sine one (1). Indeed we will find they have much in common, and that the random model allows us to reconstruct exactly the most part of the phase diagram. As before we define the Hamiltonian (here 'O' stands for orthogonal)

$$H_{\mathrm{O}} \equiv \sum_{x,y=1}^{N} O_{x,y}\sigma_x\sigma_x \tag{13}$$

where now $O_{x,y}$ is a generic orthogonal symmetric matrix. The same behaviour of the deterministic model will be obtained by using a rescaled Hamiltonian

$$\tilde{H}_{\mathrm{O}} \equiv 2N - 2H_{\mathrm{O}}\,. \tag{14}$$

The form we have just written is important since in the case of the original sine and cosine models the Hamiltonians defined after (12) can also be written in the form $2N - 2\sum_{i,k} M_{i,k}\sigma_i\sigma_k$, by neglecting terms which are irrelevant in the $N \to \infty$ thermodynamic limit.

The first element for the comparison of the two classes of models, the sine and cosine versus the random one, can be obtained by noticing general features of the high-temperature expansions of the models. For both class of models the couplings† are of order $N^{-1/2}$. The diagrams which contribute to the infinite-volume limit have the same topology for the two classes of models, and they only depend on quantities like the trace of the couplings to positive powers, which have been built to be equal in the two classes of models.

The reasoning of the former paragraph proves that the sine and cosine models defined from (1) and (3), and the model with quenched disorder defined from (14), have the same high-temperature expansion. On the other hand we have exhibited a ground state of the deterministic system which exists for prime values of $(2N + 1)$. Such a construction obviously does not apply to the disordered models. This implies that the static properties of the two classes of models (for prime values of $(2N + 1)$) cannot coincide all the way down to $T = 0$. There is a crystallization transition only in the deterministic models, thanks to the very peculiar cardinality properties of N.

We will give evidence that the random and the deterministic model do coincide at all temperatures in the metastable phase. This is the case for generic values of N, since, as we have already stressed, the cardinality of $2N + 1$ is irrelevant for the behaviour of the deterministic model in the metastable phase. A similar pattern could hold for the low autocorrelation model, but in the present case of the 2-spin interaction the analysis is far simpler, and we are able to carry it through all the way.

4. The replica approach

By using replica-theory techniques [7, 8] we will now solve the model with quenched disorder defined by the Hamiltonian (13). As usual we define the free energy of n replicas as

$$f^{(n)}(\beta) \equiv \lim_{N\to\infty}\left(-\frac{1}{\beta N}\frac{\overline{Z_O(\beta)^n} - 1}{n}\right) \tag{15}$$

where with the bar we denote the average over the quenched disorder and

$$Z_O^n \equiv \sum_{\{\sigma^a\}} \exp\left\{-\beta\sum_{a=1}^{n} H_O^a\right\}. \tag{16}$$

We have to average over the quenched disorder. To this end we have to compute

$$\overline{Z_O^n} = \int \mathrm{d}O\, \exp\left\{\sum_{k,j=1}^{N} \beta\Omega_{k,j} O_{k,j}\right\} \tag{17}$$

where the integral runs over orthogonal symmetric matrices, and

$$\Omega_{k,j} \equiv \sum_{a=1}^{n} \sigma_k^a \sigma_j^a . \tag{18}$$

We will now show that we can solve a more general problem considering a symmetric coupling matrix with some quite general pre-assigned eigenvalue distribution. We will

† This is not true for all soluble spin-glass models. In the dilute models the average coordination number z remains finite and the couplings may be a quantity of order 1, with a probability of order z/N.

derive such a more general form. We will eventually obtain the relevant result specializing this general form to orthogonal symmetric matrices.

A generic real symmetric matrix O can be decomposed as†

$$O = VDV^* \tag{19}$$

where D is a diagonal matrix which controls the spectrum of O, and V is the orthogonal matrix which diagonalizes O. By using this decomposition we have to compute

$$\overline{Z_O^n} = \int \mathrm{d}V \exp\{\mathrm{Tr}\,(\beta V \Omega_{k,j} V^* D)\} \tag{20}$$

where D is a diagonal matrix, $\mathrm{d}V$ is the Haar invariant measure over the orthogonal group, and the matrix Ω is defined in (18). We can use the results derived in [12] for unitary matrices and adapt them to the orthogonal case. So, let us assume for a while that we are integrating over unitary matrices V. Using the fact that Ω *has finite rank* we find that

$$\int \mathrm{d}V \exp\{\mathrm{Tr}\,(\beta V \Omega_{k,j} V^* D)\} = \exp\left\{N \mathrm{Tr}\, G_D\left(\frac{\beta\Omega}{N}\right)\right\}. \tag{21}$$

The value of G is given in [12] (when, as we have already said, the integral is over the unitary matrices). Following [12] let us define the generating function for the traces of D as

$$\Phi_D(j) \equiv \frac{1}{N} \sum_{k=0}^{\infty} j^k \mathrm{Tr}\, D^k \tag{22}$$

in the case where $d \equiv \mathrm{Tr}\, D = 0$. If $d \neq 0$ we define the generating functional as

$$\Phi_D(j) \equiv \frac{1}{N} \sum_{k=0}^{\infty} j^k \mathrm{Tr}\,(D - d)^k \tag{23}$$

which allows a straightforward generalization of the computation, by only adding an additional contribution to the free energy. We define the function z_D as

$$z_D(j) \equiv j\Phi_D(j) \tag{24}$$

and finally we define the function $\psi_D(z)$ by

$$\psi_D(z) \equiv \Phi(j_D(z)) \tag{25}$$

where $j_D(z)$ is obtained by inverting (24). All said, [12] tells us that G is given by

$$G_D(z) = \int_0^1 \mathrm{d}t\, \frac{\psi_D(zt) - 1}{t}. \tag{26}$$

In the orthogonal symmetric case $O^2 = 1$ and the eigenvalues of D can take the values ± 1. As far as our problem is concerned we are interested in the case where half of the eigenvalues take the value $+1$ and half the value -1. Here we will discuss a more general case, where a fraction ν of the eigenvalues is $+1$ and a fraction $1 - \nu$ is -1.

It is interesting to notice that the ground state of the model has a simple geometrical significance. Let us consider our series of N spins σ, and look at it as one of the vertices of the unit hypercube in N dimensions. Let us imagine such an hypercube as embedded in $\mathcal{R}^N$. Now we extract a random linear subspace F of dimension νN, which includes the origin. For example, if we have $N = 2$ spins the configuration will sit on one of the four

† We like to stress with * the operation of Hermitian conjugation, which for real matrices coincides with transposition.

corners of a two-dimensional square, and for $\nu = \frac{1}{2}$ we would pick a random line passing through the origin. If P is the projector of F, the matrix O is given by

$$O = 2P - 1 . \tag{27}$$

We define the norm of the projection of a spin configuration $\{\sigma\}$ over the subspace F by

$$P_\sigma = |P\sigma| \tag{28}$$

and the norm of the projection over the complementary subspace $F^\perp$ by

$$D_\sigma = |(1 - P)\sigma| \tag{29}$$

where D_σ can be interpreted as the distance of the configuration σ from the subspace F. The relation $P_\sigma^2 + D_\sigma^2 = 1$ holds. The Hamiltonian (14) can now be written as $4D_\sigma^2$. The ground-state energy is given by the minimum distance D_m of one of the 2^N configuration from the random subspace. This problem is well studied in the case $\nu N = 1$, i.e. in the limit $\nu \to 0$, mainly for its applications to perceptrons [11], but it has not been discussed in the most general case.

For $\nu = \frac{1}{2}$, by inverting the second relation, after some algebra we find (we omit the suffix $\nu = \frac{1}{2}$ for G and ψ)

$$G(z) = \int_0^1 \mathrm{d}t \, \frac{\sqrt{1 + 4z^2t^2} - 1}{2z} \tag{30}$$

which gives

$$G'(z) = \frac{\psi(z) - 1}{z} . \tag{31}$$

After integrating the last relation with the condition $G(0) = 0$ we find

$$G(z) = \tfrac{1}{2}\log(\sqrt{1 + 4z^2} - 1) - \tfrac{1}{2}\log(2z^2) + \tfrac{1}{2}\sqrt{1 + 4z^2} - \tfrac{1}{2} \tag{32}$$

where the constant term has be chosen such that $G(0) = 0$.

We have already said that we have obtained this G for V unitary. It is easy to argue that when we integrate over orthogonal matrices the only difference is that $G(\beta z)$ gets substituted from $\frac{1}{2}G(2\beta z)$. This can be seen, for example, by noticing that the function G has to be the same in the two cases (since the same diagrams contribute) and at first order in β orthogonal and unitary matrices have to give the same results. So the only allowed renormalization will be of the kind $G(z) \to \alpha G(z/\alpha)$. The counting of the eigenvalues leads to the conclusion $\alpha = \frac{1}{2}$.

Using the fact that for integer positive k

$$\mathrm{Tr}\left(\frac{\beta\Omega}{N}\right)^k = \mathrm{Tr}\,(\beta\Sigma)^k \tag{33}$$

where the matrix Σ is defined as

$$\Sigma_{a,b} \equiv \sum_{k=1}^{N} \sigma_k^a \sigma_k^b \tag{34}$$

it follows immediately that

$$\mathrm{Tr}\, G\left(\frac{\beta\Omega}{N}\right) = \mathrm{Tr}\, G(\beta\Sigma) . \tag{35}$$

To continue our computation we insert a δ-function, and introduce the Lagrange multipliers Λ with the representation

$$\prod_{a,b=1}^{n} \delta\Big(\sum_{j=1,N} \sigma_j^a \sigma_j^b - N Q_{a,b} \Big) \simeq \int \prod_{a,b=1}^{n} \mathrm{d}\Lambda_{a,b} \exp\left\{ \mathrm{i} \sum_{a,b} \Lambda_{a,b} \Big(\sum_{j=1,N} \sigma_j^a \sigma_j^b - N Q_{a,b} \Big) \right\}. \tag{36}$$

After a little more algebra (very similar to the one developed in I) we find that

$$\overline{Z^n} = \int \mathrm{d}Q \, \mathrm{d}\Lambda \, \exp(-N A[Q, \Lambda]) . \tag{37}$$

In the large-N limit the free energy is obtained by finding the saddle-point value of $A[Q, \Lambda]$, which has the form

$$A[Q, \Lambda] = -\tfrac{1}{2} \mathrm{Tr}\, G(2\beta Q) + \mathrm{Tr}\,(\Lambda Q) - F(\Lambda) \tag{38}$$

where G has been already defined, and

$$F(\Lambda) \equiv \ln \sum_{\sigma^a} \exp\left\{ \sum_{a,b} \Lambda_{a,b} \sigma^a \sigma^b \right\}. \tag{39}$$

We will need to study (38) to discuss the solutions of the model.

5. Saddle-point equations and replica-symmetry breaking

In the previous section we have found the saddle-point equations which allow us to solve the model with quenched disorder defined in (13). Let us recall that the free energy (multiplied by $n\beta$) in terms of the matrices Q and Λ is

$$\beta f_{\mathrm{O}} = \lim_{n \to 0} \frac{A[Q_{\mathrm{SP}}, \Lambda_{\mathrm{SP}}]}{n} \tag{40}$$

where A is defined in (38), and Q_{SP} and Λ_{SP} are evaluated at the saddle point of A.

The free energy (14) of the model with quenched random disorder (which has the same high-temperature expansion as the deterministic one (1) and (3)) is given by

$$\beta f = 2\beta - 2\beta f_{\mathrm{O}}(2\beta) . \tag{41}$$

Let us start by considering the annealed case, $n = 1$. Here the matrix Σ is set equal to 1. The action does not depend on Λ, and we find for the free-energy density and the internal energy

$$\begin{aligned} f &= 2 - \frac{1}{2\beta} G(4\beta) - \frac{1}{\beta} \log(2) \\ e &= 2(1 - G'(4\beta)) = 2 - \frac{\sqrt{1 + 64\beta^2} - 1}{4\beta} . \end{aligned} \tag{42}$$

We plot the replica-symmetric free energy found in (42) in figure 1 (together with the one-step replica-broken result we will compute in the following). In figure 2 we plot the internal energy and in figure 3 the entropy of the system.

In the high-temperature region the quenched and the annealed solutions coincide as usual for long-range models.

The replica-symmetric solution is stable at all temperatures. But, since for $T \leqslant 0.26$ it gives a negative entropy (see figure 3), it cannot be correct down to $T = 0$. We expect replica symmetry to break above (but very close) to $T = 0.26$. Here the system enters a

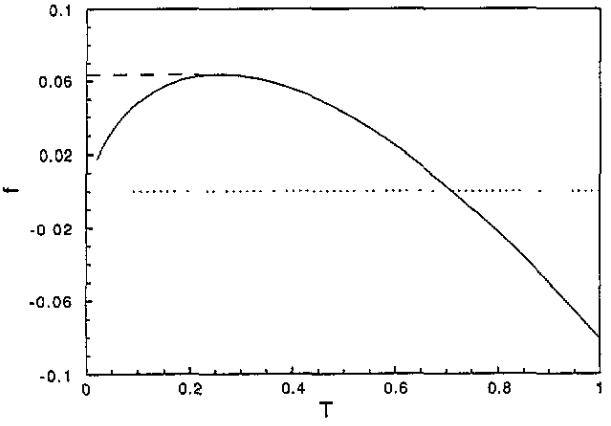

Figure 1. Free energy of the model with quenched random disorder versus T. *The full curve is the replica-symmetric* solution, the broken curve is the one-step replica-broken solution. With the dotted curve we indicate the zero of the free energy. The free energy vanishes at $T \sim 0.71$.

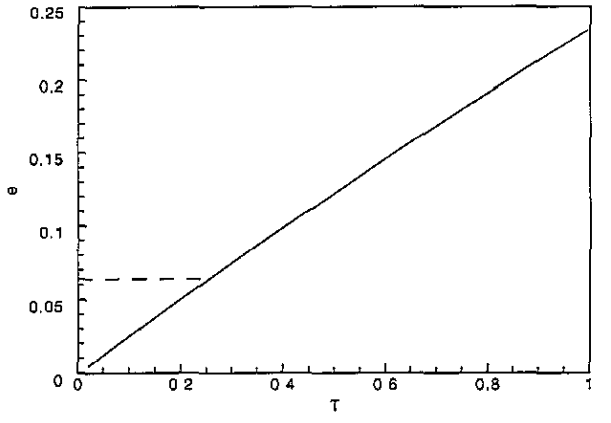

Figure 2. As in figure 1, but for the internal energy of the system.

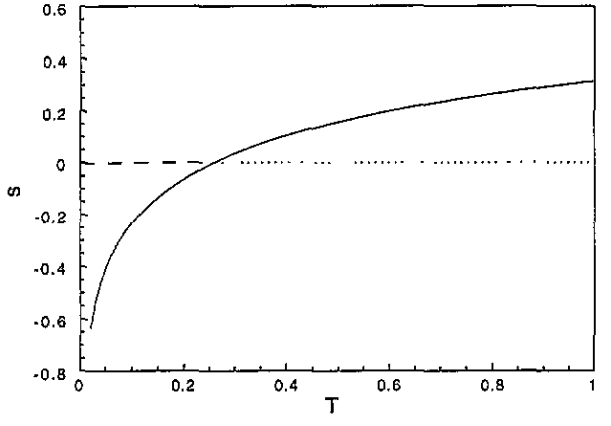

Figure 3. As in figure 1, but for the *entropy of the system. Here again the* dotted curve is solely meant to indicate the zero. The entropy of the one-step solution is very small in the low-T phase.

glassy phase very similar to that of the random energy model [13] and of the p-spin systems (see, for example, [14–17]).

We can compute the one-step replica-broken solution. We parametrize the matrices Q and Λ in the usual way. In the presence of a uniform magnetic field the matrix elements Q_{ab} take the value q if a and b belong to the same sub-block of size m, while they take

the value q_0 if they do belong to different sub-blocks. We parametrize the matrix Λ_{ab} with blocks of the same size m, and we set its elements equal to λ or λ_0 with the same procedure we used for Q. Here we consider the simpler case of zero magnetic field, where the parameters q_0 and λ_0 are zero and we set

$$Q_{a,b} = q \quad (a \neq b) \qquad \Lambda_{a,b} = \lambda \quad (a \neq b) \tag{43}$$

inside the blocks of size m ($Q_{aa} = 1$; $\Lambda_{aa} = 0$). After some algebra we obtain

$$\beta f = 2\beta - \frac{1}{2m}\left[(m-1)G(4\beta(1-q)) + G(4\beta(mq+1-q))\right] + \lambda q(m-1) - \log(2) + \lambda - \frac{1}{m}\log\int_{-\infty}^{\infty}\frac{dx}{\sqrt{2\pi}}e^{-x^2/2}\cosh^m(\sqrt{2\lambda}\,x). \tag{44}$$

The stationary equation for q tells us that

$$\lambda = \frac{2\beta}{m}\left[G'(4\beta(mq+1-q)) - G'(4\beta(1-q))\right]. \tag{45}$$

We can use this relation to eliminate λ from (44). We are left with a function of q and m, and we have to find a stationary point. This expression cannot be solved in close form. We have plotted the numerical solution as broken curves in figures 1–3.

At $T_{RSB} \sim 0.26$ there is a phase transition to a phase with broken replica symmetry. At the transition point T_{RSB} the value of the entropy is finite but very small (~ 0.0004), the value of q jumps discontinuously to a value very close to 1 (~ 0.9998), and λ is large but finite (~ 10) (in the random energy model at the transition point $q = 1$ and $\lambda = \infty$ [18]). Below T_{RSB} the parameter m is very approximately proportional to T, $m = 1$ at T_{RSB}. This is the typical scenario for a large class of models where the order parameter jumps discontinuously at the transition.

We have not studied the stability properties of the replica-broken solution in detail. It is possible that the one-step solution is stable down to a very low temperature, and that for lower values of T a continuous symmetry breaking is needed to describe the system. This is what happens for the p-spin model [19]. As we will discuss in the next sections this second transition would probably have no relevance from the physical point of view, since the system is not able to explore the lowest free-energy configurations. We will see that in a usual annealing process (i.e. a slow temperature cooling starting from a high temperature) the system has a transition at a temperature T_G well above the temperature T_{RSB} where replica symmetry breaks down. We will name the transition at T_G the *glass transition*. This transition is dynamical in nature and corresponds to the presence of a very large number of metastable states. At T_G the system remains trapped in a metastable state, and thermal fluctuations are very small.

6. The marginality condition

In the framework of mean-field theory it has been suggested that there are models (in particular the p-spin spherical spin-glass model) in which, below a temperature T_G, the Sompolinsky–Zippelius equilibrium dynamical equations [20, 14, 15] admit a new solution whose energy does not coincide with the one computed in the static approach [16]. It is tempting to conjecture that this solution describes metastable states, which only exist below T_G.

This analysis is confirmed by the study of the off-equilibrium dynamics. Indeed, in the case of the p-spin spherical spin-glass model it is possible to write closed equations for the correlation and response functions in the off-equilibrium regime [17]. It has been noted

that these dynamical equations also undergo a glass transition at the same temperature T_G where the equilibrium relaxational dynamics slows down. The temperature T_G is larger than the transition point where, according to the static approach, replica symmetry breaks down. Below the temperature T_G the time-homogeneity hypothesis for the correlation functions fails and aging effects start to appear. Also the standard fluctuation–dissipation theorems are not valid here.

In the SK model the temperature T_G where aging effects appear coincides with the transition point derived in the static approach, where the replica-symmetric solution becomes unstable [21]. It has also been shown that in this case the dynamic energy coincides with the static energy implying the absence of metastable states. It would seem that a dynamical transition only exists in those models in which, in the static approach, the spin-glass order parameter $q(x)$ is a discontinuous function of x.

It has also been noticed that in these models the temperature T_G coincides with the temperature T_{MC}, which is characterized as follows. Below T_{MC} a solution to the mean-field equations exists with the replica symmetry broken at one step where the size of the replica matrix sub-block m is not fixed variationally (as in the usual static approach) but is fixed by the condition that the replicon eigenvalue vanishes. This has been called the *marginality condition* [22, 16, 17], which is similar to the marginality condition of the statics [23], which is implemented in models where the function $q(x)$ is a non-constant continuous function of x. Very probably the marginality condition gives a good estimate of T_G because of the coincidence with the condition of first occurrence of an exponentially large number of extrema of the free energy. One may hope that for a wide class of models the temperature T_G coincides with the temperature T_{MC}; this conjecture is still unproven in the general case.

The models we are describing in this work (the model with quenched random disorder as well as the deterministic one) are good candidates for a test of the marginality condition principle. One can do that by comparing the theoretical prediction for T_{MC} with the output of numerical simulations. The main reason for that to happen is that at the transition point the order parameter q jumps discontinuously to a value extremely close to 1 while in other systems (as in the $p = 3$ p-spin spherical model) it is of order of 0.5. In our case the fluctuations in the low-temperature phase are practically forbidden. The system essentially freezes. As we shall see later the difference between the static transition temperature T_{RSB} and the dynamical transition-temperature value T_G is large. In the following sections we will use numerical simulations to show that, for reasons not completely clear to us, the principle seems to also work well in this case.

Now we want to compute the value of T_{MC} in our particular case. We start from (38) and we compute the Hessian matrix in the Λ, Q space. The interested reader can find the technical details in the appendix. The marginality condition gives

$$16\beta^2 G''(4\beta(1-q))\langle\cosh(\sqrt{2\lambda}x)^{-4}\rangle = 1 \tag{46}$$

where the expected value is defined by

$$\langle A(x)\rangle = \frac{\int dx\,(e^{-x^2}/\sqrt{2\pi})\cosh^m(\sqrt{2\lambda}x)A(x)}{\int dx\,(e^{-x^2}/\sqrt{2\pi})\cosh^m(\sqrt{2\lambda}x)}. \tag{47}$$

We can find the dynamical transition point by maximizing the free energy under the marginality condition.

Maximizing the free energy (44) as a function of q for m fixed, under condition (45), we find that there are values of $m \leqslant 1$ such that (46) is satisfied as soon as $T \leqslant T_{MC} \simeq 0.535 \pm 0.005$. This transition temperature is about two times larger than T_{RSB}. We also get m and q as a function of the temperature. At T_{MC} q jumps discontinuously to

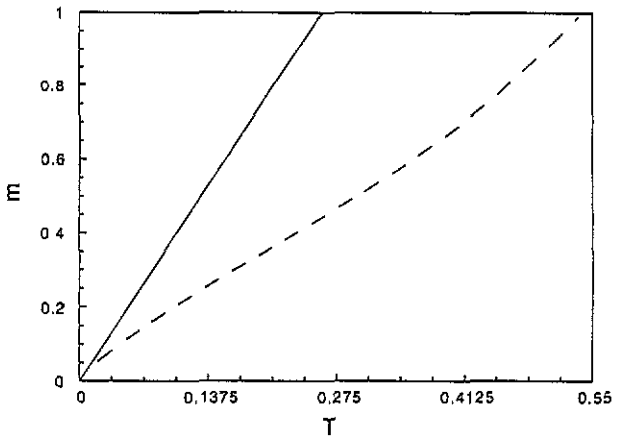

Figure 4. Replica matrix sub-block size m as a function of T. The full curve is for the static value, the broken curve is for the solution satisfying the marginality condition.

a value $\simeq 0.962$. This value is smaller than the value we have found for the static solution at the temperature T_{RSB}.

The reader should notice that we cannot expect the free energy derived using the marginality condition principle at a temperature smaller than T_{MC} to be a reasonable quantity, i.e. to satisfy the main inequalities of the thermodynamics. This is because we are in the wrong branch of the solutions of the replica equations, and we have not chosen m following a variational principle. For example, the relation $u = \partial(\beta f)/\partial\beta$ is not satisfied for the marginality condition free energy. It may be interesting to note that the value of the break-point parameter m which we have found by imposing the marginal conditions at temperatures lower that T_{MC} (which we plot in figure 4 together with the value from the static result) is not proportional to T at low temperatures.

At the present moment, using only the information from this modified static approach it is not possible to compute the behaviour of the system in the whole glassy phase, except for the value of the glassy temperature. It is not clear if the analysis should be confined to the case of a single-step replica-symmetry breaking step. It would be very interesting to analyse the full low-T implications of the marginality condition for a larger number of breaking steps, and eventually for a continuous breaking pattern. It also possible that there are no shortcuts, and that to get the full picture one has to solve the full off-equilibrium equations, which should be established by using techinques similar to the ones developed in [17, 21].

In the following sections we will present a numerical study of the model with quenched disorder and of the deterministic model. We will see that in both cases the system undergoes a dynamical transition at T_{G}, and that T_{G} is very close to the value T_{MC} we have computed here.

7. Numerical simulations of the disordered model

The model with quenched disorder is based on symmetric orthogonal interaction matrices. In order to produce the interaction matrices needed in our simulations we started by generating a symmetric matrix with random elements with a Gaussian distribution. Starting from such a matrix we have obtained a symmetric orthogonal matrix by using the Graham–Schmidt

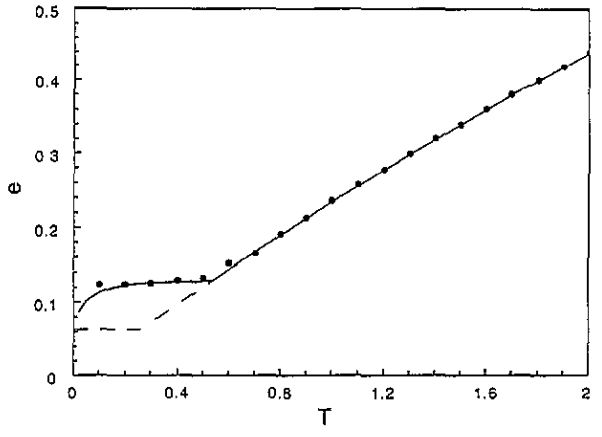

Figure 5. Energy of the model based on random orthogonal matrices versus T for $N = 186$. The broken curve is the static one-step replica-broken solution. The full curve is the prediction of (46).

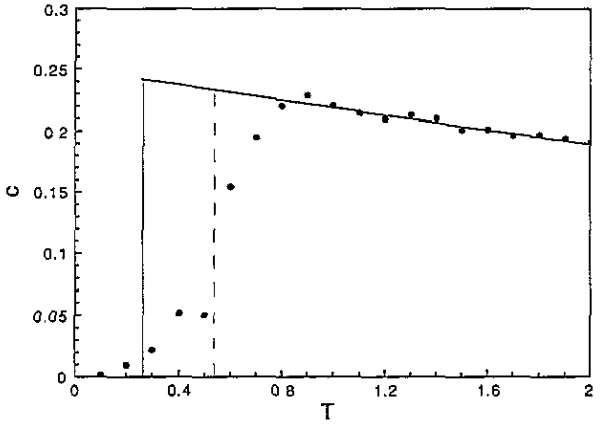

Figure 6. As in figure 5, but for the *specific heat. Here the full curve is the* static one-step replica-broken solution, while the broken curve is the prediction of (46) (inverting the notation of the former figure).

orthogonalization procedure.

Such a model has an infinite-range interaction, and Monte Carlo simulations are quite time consuming (but much less time consuming than, for example, p-spin models with $p > 2$). With limited computer time (on a reasonable workstation time allocation) we have been able to obtain reliable results for samples with a volume of up to a few hundred spins.

In figure 5 we show our estimate for the internal energy on one disorder sample, for $N = 186$. In figure 6 we show the specific heat. We have started the run from high T and we have been decreasing the temperature in steps of 0.1.

We have tested that sample to sample fluctuations and finite-size corrections in the internal energy and heat capacity are negligible.

Our numerical results fit the theoretical predictions well for temperatures larger than $T_G \sim 0.5$. At T_G the system freezes. The energy does not decrease further than a value close to 0.12 and the specific heat decreases to a very small value. This is the dynamical transition we discussed in the previous section. T_G is well above the temperature T_{RSB} and coincides with the transition point derived for the marginality condition.

The transition at T_G is of a dynamical nature. The system does not reach the lowest-lying states (which have an energy close to 0.063). One could doubt if the freezing at $T_G \sim 0.5$ is a finite time effect. In figure 7 we show the internal energy of the system as a function of T (here $N = 100$; we have used a value of N not too small in order to make

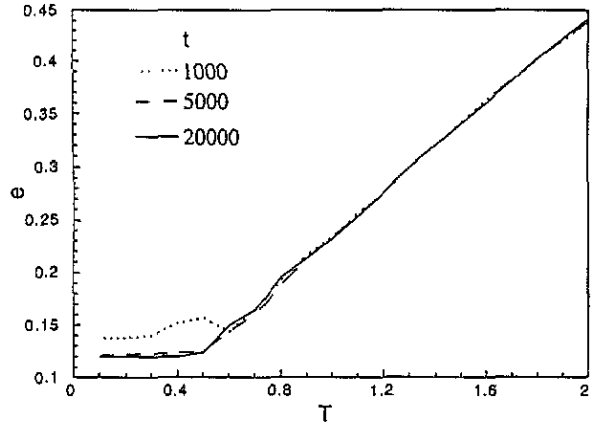

Figure 7. Energy of the model with quenched disorder versus T for $N = 100$. Different curves correspond to different annealing schedules.

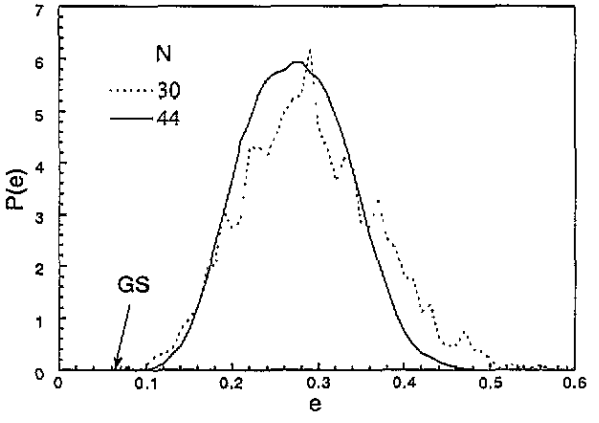

Figure 8. Probability distribution of the energy of the metastable states for the sizes $N = 30$ and 44 for the model with quenched disorder. The correct ground-state energy is indicated with an arrow.

the metastability visible). We plot three different curves for different run lengths. In the run with $t = 1000$, for example, we sweep the lattice 1000 times at each T point during our annealing procedure (i.e. while systematically decreasing T).

When the annealing time is too short for $T < T_G$ we get an energy that is too high. But as soon as the scheduling becomes slow enough we see that the energy thermalizes. The dynamical freezing appears to be a genuine behaviour which survives in the limit of infinite times for large volumes. Let us note that for sizes less than $N \sim 50$ the system is able to find the ground state in a reasonable time on our simulation time-scale, and we see it leaving the glassy phase. The limits $N \to \infty$ and $t \to \infty$ seem not to commute.

Finally, in figure 8 we show the distribution probability for the energy of the metastable states at zero temperature for quite small system size (where we are able to reach the true ground state of the system).

For each lattice volume we ran several million Monte Carlo runs at zero temperature (we sweep the lattice sequentially and we flip the local spin if in so doing the internal energy decreases) starting from different initial conditions and searching for metastable states. We stop the search after we have found the lowest energy state 100 times. We take that as good evidence for having collected a fair sample of the low-lying states. In figure 8 we have also drawn an arrow locating the ground-state energy given by (44) (which is close to 0.063). The agreement with our zero-temperature results is good. We also see that the distribution shape of the metastable states is reminiscent of that found in the case of the SK model [24].

We will also see in the following that the energy distribution for the deterministic model is similar to the one of the model with quenched disorder (except for the existence of the very low-lying ground state we have written explicitly for certain values of N).

8. Numerical simulations of the deterministic model

We have studied the *cosine* model by using numerical simulations. We will start by presenting results which describe the nature of the ground state and illustrate the existence of a crystallization transition for values of N such that $(2N+1)$ is prime. Then we will discuss the behaviour of the internal energy and of the specific heat during an annealing process.

As we have discussed in section 2, the Hamiltonian (12) admits a zero-energy ground state for values of N such that $(2N+1)$ is prime. We have found the ground state by exact enumeration for small N values (see I for a detailed discussion of the technique). For higher values of N we have found the ground state by looking for solutions of the naive mean-field equations, as we describe in the next section. For finding the ground state this method is slightly more efficient than the zero-temperature Monte Carlo method introduced in the previous section. In figure 9 we plot the ground-state energy divided by N versus N^{-1} for different values of N (at $N=\infty$ we plot the one-step replica-broken analytic result we have obtained for the ground state of the model with quenched disorder). For N such that $(2N+1)$ is prime we also plot, with a different symbol, the energy divided by N of the first excited state. The energy per spin is of order 0.1. The data of figure 9 appear to be good evidence that for generic values of N the ground-state energy tends to the value computed by the replica approach (we suggest the curious reader compare these results with those of I, since the difference is easy to appreciate), and that the energy density does not vanish in

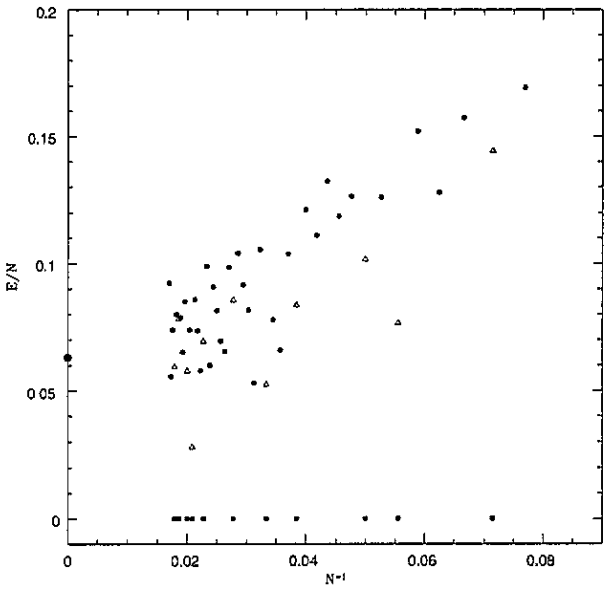

Figure 9. The ground-state energy of the sine model divided by N versus N^{-1} for small values of N (at $N=\infty$ we plot the one-step replica-broken analytic result we have obtained for the ground state of the model with quenched disorder). For $(2N+1)$ prime we also plot the energy of the first excited state with open triangles.

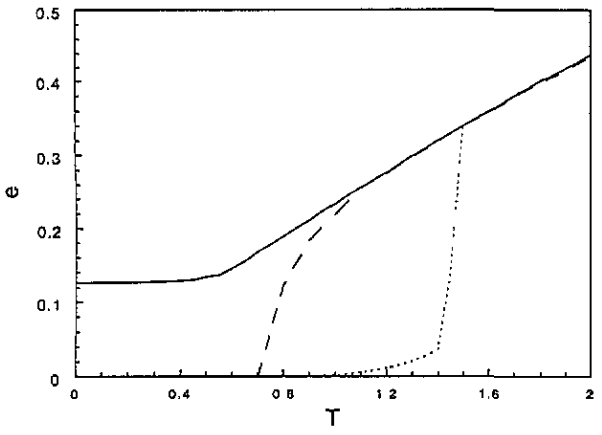

Figure 10. Energy of the cosine model versus T, for prime values of $(2N+1)$. The full curve is for decreasing T starting from a random configuration (and is the same for the two lattice sizes), while the broken curve starts from a true ground state for increasing T, $N = 44$, and the dotted curve is for increasing T, $N = 806$.

the thermodynamic limit. The excited states for $(2N+1)$ prime are a bit lower than the ground state for generic N values, but they do not seem to have an atypical behaviour. In other words it would seem clear that the pathology of the prime values $(2N+1)$ is confined to the ground state. The spectrum of the higher-energy states, including the first excited state, does not depend on the cardinality of $(2N+1)$.

For prime values of $(2N+1)$ we find a crystallization first-order transition for $T_C \sim 0.7$. Knowing the exact form of the ground state for such N values has been a remarkable plus, which allows us to study the system both starting from high T and cooling down to low T (in this case the system never does find the true ground states, but gets trapped at the energy of the metastable phase) and starting *from the ground-state configuration*, slowly increasing the temperature T. In this way we are able to observe a thermal cycle we would not be able to detect in any other way. We show the results (for $N = 44$ and $N = 806$, both such that $(2N+1)$ is prime) in figure 10. The full curve is for decreasing T (and is the same for the two lattice sizes), while the broken curve is for increasing T, $N = 44$, and the dotted curve is for increasing T, $N = 806$.

We notice that the area included between the ascending and descending curves increases with increasing N. The crystallization transition is of first order, since the energy and the entropy are discontinuous at T_C. The discontinuities ΔE and ΔS are such that $\Delta E = T_C \Delta S$. The free energy vanishes at approximately T_C (see figure 1) and remains very close to zero below T_C in the crystalline phase. In fact, at low temperatures the energy needed for a spin-flip starting from the ordered ground state is in the range 6–10 so that the parameter for a low-temperature expansion of the free energy is of the order of $\exp\{-6/T\}$. This means that the low-temperature expansion is well convergent and has a free energy which differs from zero by a rather small amount in the whole region $T < T_C$. The high-temperature free energy (given by (42)) and the low-temperature free energy (which is equal to zero) intersect with an angle which is in agreement with the first-order nature of the crystallization transition.

Dynamically, our system is only able to undergo a crystallization transition for small values of N which satisfy the cardinality condition. If $(2N+1)$ is prime and N is very large a local Monte Carlo annealing dynamics is unable to bring the system to its true ground state. The system remains in a metastable phase exactly as it does in the model with quenched disorder (where the zero-energy ground state does not exist). In this regime the cardinality condition is irrelevant. This is illustrated by figures 11 and 12. We plot the energy and the specific heat versus T for the cosine model and for the model with quenched disorder (from numerical simulations), for the one-step broken solution and for

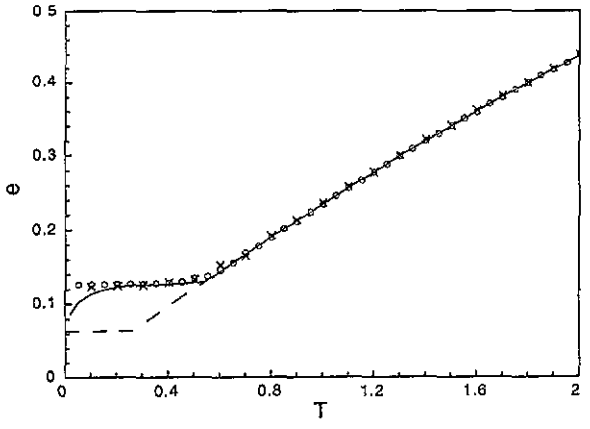

Figure 11. The internal energy as a function of T. The open circles are from numerical simulations for the cosine model, $N = 806$; crosses from numerical simulations for the model with quenched disorder, $N = 186$. The broken curve is for the static one-step replica-symmetry broken solution, the full curve for the result obtained by imposing the marginality condition.

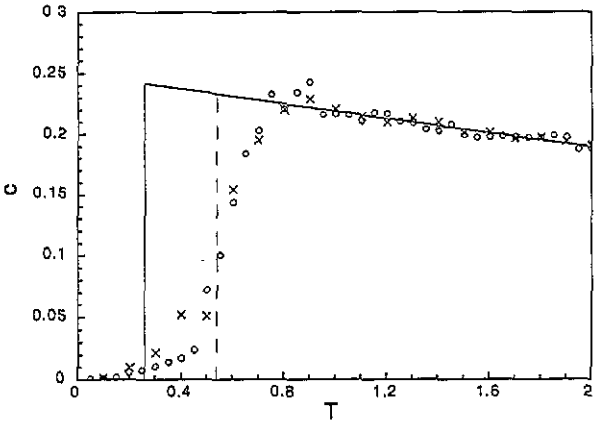

Figure 12. As in figure 11, but for the specific heat.

the marginality condition solution.

The model with quenched disorder has been conceived in order to reproduce the high-T expansion of the deterministic model. Below the glass temperature T_G there are no *a priori* reasons why the two models should behave in a similar way. The fact that the two models also coincide in the metastable phase is clear from the results we show in figures 11 and 12, and comes as a very nice surprise. One of the reasons for such behaviour is the fact that the metastable states in the two models have a very similar distribution, as we will show more fully in the next section.

Also in the case of the deterministic model, the metastable phase can be described using the marginality condition (46) of section 6.

Figure 11 shows that the solution where the marginality condition has been imposed describes the numerical results very well down to $T \sim 0.1$. Below that temperature the energy of the analytic solution departs from the numerical results reaching the static value ~ 0.063 at $T = 0$. This behaviour is related to the fact that the break-point parameter m (as determined by imposing the marginality condition) is not proportional to T for low values of T. This fact will be discussed in more detail in the appendix and confirms the fact that the static replica equations are useful for predicting the existence of the glassy transition at T_G but possibly not the full low-T region.

The next section is devoted to describing the structure of the metastable states for the deterministic model at zero temperature by analysing the numerical solution of the naive

TAP equations. But for the existence of a crystalline state in the case of $(2N + 1)$ prime the shape of the distribution of the metastable states will be shown to be similar to the one found in case of the random model.

9. Mean-field equations for the deterministic model

The naive mean-field equations for the sine model can be defined through the iterative relation

$$m_x = \tanh\left(\beta \sum_{y \neq x} S_x(m_y)\right) \tag{48}$$

where the function S_x has been defined in (2). Obviously we could have defined the analogous equations by using the C function defined in (4).

We are interested in the low-temperature limit of the model. We can thus avoid considering the complete TAP equations, where the reaction field is included, which are far more difficult to deal with. In the low-temperature limit we can solve the even simpler equations

$$m_x = \text{sign}(h_x) \tag{49}$$

where h_x is the local field acting on the spin x.

We find the $T = 0$ solution of these equation by cooling the solution found at $T > 0$. In figure 13 we show the number of solutions of a given energy as a function of the energy for a typical prime (broken curve) and non-prime value (full curve) of $p = (2N + 1)$, respectively.

Analogously to I we stop our ground-state search after finding the states with the lowest energies five times, which makes us confident that we have sampled the low-energy states

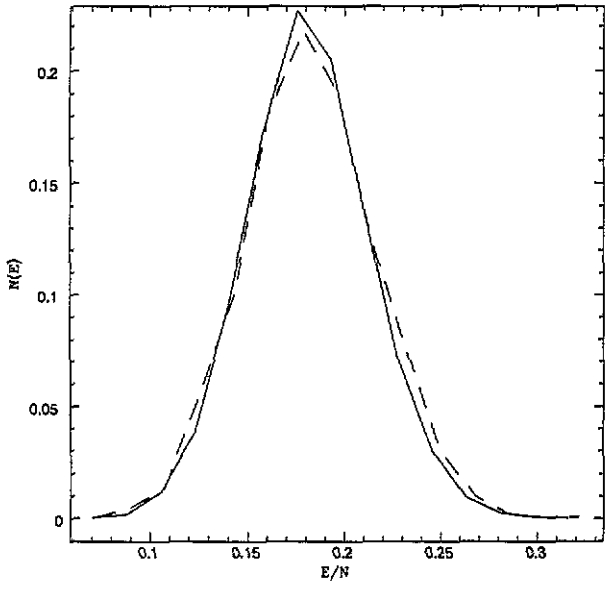

Figure 13. The number of solutions of the $T = 0$ mean-field equations of a given energy as function of the energy for $N = 56$, where $(2N + 1)$ is prime (broken curve) and for $N = 57$, where $(2N + 1)$ is not prime (full curve).

with good accuracy. We have studied systems with N up to 64. For prime values of $(2N+1)$, where we know the exact ground state, this method has always found the correct ground-state energy (i.e. zero).

10. Conclusions

Building upon the idea introduced in paper I, here we have introduced a class of deterministic spin models which do not contain disorder, but whose low-T behaviour is dictated by self-induced frustration. They are potentially relevant to the description of the glass state. Using number theory we have been able to exhibit a zero-energy ground state for given values of the volume N (such that $(2N+1)$ is prime).

We have proceeded by writing a model with quenched random disorder, based on orthogonal interaction matrices, which reproduce the high-temperature expansion of the deterministic models. By using replica theory and well known results of integration on Lie groups we have been able to solve the model with quenched disorder. The model with quenched disorder has a replica-symmetry breaking transition at a quite low temperature. The phase transition is discontinuous, as in the random energy model.

We have also studied the low-T phase. Even if the random model does not coincide with the deterministic model for all values of N down to $T = 0$ (since we know that for prime values of $(2N+1)$ the deterministic model admits a ground state based on Legendre sequences which we cannot find in the random approach) we have found that in all of the metastable phase the two classes of models coincide. We have also found, remarkably, that for generic values of N even the ground states of the models seem to coincide (as from figure 9).

We have shown that, for the values of N which satisfy the cardinality condition, the deterministic model undergoes a crystallization transition. This transition is of the first order, from the thermodynamical point of view, since the energy and the entropy jump discontinuously. Even if we cannot be sure of this fact, our exact solutions of small systems give a precise hint favouring the absence of a zero-energy ground state for generic N values.

We have shown that the structure of metastable states of the two classes of models has much in common (for this effect the cardinality of N is irrelevant). For the model with quenched disorder we have performed Monte Carlo runs at zero temperature searching for locally stable states. In the deterministic case we have solved the naive TAP equations. The similarity of the shapes of the distribution of metastable states suggests that the dynamical behaviour of the two models must be very similar. The two figures 11 and 12 are quite decisive in this respect. The two models behave very similarly—they both display a singularity at a temperature T_G where the system freezes and thermodynamic fluctuations (related, for example, to the specific heat and to the magnetic susceptibility) vanish. We have also shown that, for reasons that are quite unclear to us, the marginality condition gives a good estimate of the low-T behaviour.

These results strengthen the idea that the off-equilibrium dynamics for the deterministic model should be very similar to the one of the model with quenched disorder. We would expect, for example, that the deterministic model could display aging effects like those which affect the random model and many models based on quenched disorder [25]. We have measured the usual time–time correlation function between the spin configuration at the waiting time t_w and the spin configuration at a later time $t_w + t$. We have observed that below T_G the shape of the correlation function depends on the previous history, i.e. on t_w. These results are very similar to those also found in related deterministic models like the

low autocorrelation binary sequences [4]. It seems that deterministic models also display non-equilibrium effects very similar to those of spin glasses with randomness.

We hope that the results of this paper will be relevant to a large variety of different problems in condensed matter physics, where it is natural to study systems with a complex free-energy landscape in which quenched disorder is not present as a given, pre-assigned condition.

Acknowledgments

We thank Andrea Crisanti, Leticia Cugliandolo and Jorge Kurchan for discussions. GP thanks Bernard Derrida for stressing to him the interest of computing the average minimal distance of an Ising spin configuration from a random hyperplane.

Appendix.

In this appendix we present some technical details about how we applied the marginality condition in our computation at one step of replica-symmetry breaking. Our starting point is the expression for the free energy

$$A[Q,\Lambda] = -\tfrac{1}{2}\mathrm{Tr}\, G(2\beta Q) + \mathrm{Tr}\,(\Lambda Q) - F(\Lambda) \tag{A1}$$

with $G(Q)$ given by

$$G(2\beta Q) = \sum_{k\geqslant 1} \frac{(2\beta)^{2k}}{2k}\psi_{2k}\,\mathrm{Tr}\,\mathrm{Tr}\, Q^{2k} = \sum_{k\geqslant 1} c_{2k}\,\mathrm{Tr}\, Q^{2k} \tag{A2}$$

where the ψ_{2k} are the Taylor coefficients of the series expansion of the function $\psi(z)$:

$$\psi(z) = 1 + \sum_{k\geqslant 1}\psi_{2k}\, z^{2k}\,. \tag{A3}$$

In the most general case the stability condition implies that the Hessian matrix of the second derivatives of $A[Q,\Lambda]$ in the space of matrices $\{Q,\Lambda\}$ around the equilibrium solution is negative-definite (the integration path in Λ-space runs on the imaginary axis, and the stability condition has the opposite sign to the usual case). To construct the Hessian we compute the second derivatives of (A1). This gives a four block matrix with the derivatives $\partial_{QQ}A$, $\partial_{\Lambda\Lambda}A$, $\partial_{Q\Lambda}A$ and the identical symmetric block $\partial_{\Lambda Q}A$. The sub-block $G \equiv \partial_{QQ}A$ is given by

$$G_{(ab)(cd)} = \frac{\partial^2 A}{\partial Q_{ab}\partial Q_{cd}} = \sum_{k\geqslant 1} 4kc_{2k}\frac{\partial\left(Q^{2k-1}\right)_{ab}}{\partial Q_{cd}}\,. \tag{A4}$$

The matrix G has three different types of elements, depending on whether the replica indices (ab) and (cd) do coincide, have one element in common, or are completely different. For these three different cases we have

$$\begin{aligned}
\frac{\partial\left(Q^{2k-1}\right)_{ab}}{\partial Q_{cd}} &= \sum_{p=0}^{2k-2}\left(\left(Q^{p}\right)_{ac}\left(Q^{2k-2-p}\right)_{db} + \left(Q^{p}\right)_{ad}\left(Q^{2k-2-p}\right)_{cb}\right)\\
\frac{\partial\left(Q^{2k-1}\right)_{ab}}{\partial Q_{ac}} &= \sum_{p=0}^{2k-2}\left(\left(Q^{p}\right)_{aa}\left(Q^{2k-2-p}\right)_{cb} + \left(Q^{p}\right)_{ac}\left(Q^{2k-2-p}\right)_{ab}\right)\\
\frac{\partial\left(Q^{2k-1}\right)_{ab}}{\partial Q_{ab}} &= \sum_{p=0}^{2k-2}\left(\left(Q^{p}\right)_{aa}\left(Q^{2k-2-p}\right)_{bb} + \left(Q^{p}\right)_{ab}\left(Q^{2k-2-p}\right)_{ab}\right).
\end{aligned} \tag{A5}$$

The other sub-blocks I and M are

$$I_{(ab)(cd)} = \frac{\partial^2 A}{\partial \Lambda_{ab} \partial Q_{cd}} = \delta_{(ab)(cd)}$$
$$M_{(ab)(cd)} = \frac{\partial^2 A}{\partial \Lambda_{ab} \partial \Lambda_{cd}} = \langle \sigma_a \sigma_b \rangle_F \langle \sigma_c \sigma_d \rangle_F - \langle \sigma_a \sigma_b \sigma_c \sigma_d \rangle_F . \tag{A6}$$

The mean value $\langle \cdots \rangle_F$ in the last equations is taken over the action (39). M is the usual Hessian which determines the stability of the SK model.

Now it is easy to see that for each eigenvalue of the sub-block matrices G and M, (for instance g and μ, respectively) the stability condition is determined by

$$g\mu - 1 \leqslant 0 \tag{A7}$$

with the marginal condition being the equality. We now have to compute all the eigenvalues of the matrices G and M and to search for the ones which maximize the product $g\nu$. For the p-spin model (and also the SK model) this condition is relatively easy to determine because there is a unique eigenvalue g for G (in that the case the matrix G is g times the identity matrix) and the maximum eigenvalue of M is found in the replicon sector when all replicas belong to the same block (once replica symmetry is broken).

In the present case, even though the maximum value of M is the usual one [26], G has more than one eigenvalue. We have searched for all of them in the case of one step of replica-symmetry breaking. We have evaluated the derivatives for the matrices Q and Λ broken according to the scheme of (43). The general expression for the eigenvalues at one step of replica-symmetry breaking has been given in [27]. There are two longitudinal eigenvalues, four anomalous eigenvalues and four replicons which finally reduce to only five different eigenvalues (this is because we set $Q_{(ab)} = 0$ if the indices (a, b) do not belong to the same sub-block of size m). These are given by

$$g_1 = \frac{16\beta^2}{m}(G''(4\beta(1-q)) + (m-1)G''(4\beta(mq+1-q))) \tag{A8}$$

$$g_2 = 16\beta^2 G''(4\beta(1-q+mq)) \tag{A9}$$

$$g_3 = \frac{32\beta^2}{m}G''(4\beta(1-q)) + \frac{4\beta(m-2)}{qm^2}(G'(4\beta(1-q+mq)) - G'(4\beta(1-q))) \tag{A10}$$

$$g_4 = \frac{4\beta}{qm}(G'(4\beta(1-q+mq)) - G'(4\beta(1-q))) \tag{A11}$$

$$g_5 = 16\beta^2 G''(4\beta(1-q)) . \tag{A12}$$

where g_5 is the replicon, with all the replica indices belonging to the same sub-block. Taking for the matrix M the replicon eigenvalue corresponding to the four replica indices all belonging to the same sub-block, we find

$$\mu = \langle \cosh^{-4}(\sqrt{2\lambda}x) \rangle \tag{A13}$$

where the expectation value is defined by

$$\langle A(x) \rangle = \frac{\int dx\, (e^{-x^2}/\sqrt{2\pi}) \cosh^m(\sqrt{2\lambda}x) A(x)}{\int dx\, (e^{-x^2}/\sqrt{2\pi}) \cosh^m(\sqrt{2\lambda}x)} . \tag{A14}$$

Inserting this value of μ in (A7), we have searched among the five values of g for the one which gives the maximum free energy when the stability is marginal (i.e. when (A7) is saturated). We have found that the eigenvalue g_5 is the one which gives the maximal free energy. This leads us to the marginality condition (46). We have searched for a solution

of the marginality condition in which m behaves linearly with T for low temperatures, but we have not been able to find it. It is plausible that such a well behaved solution does not exist and that to improve our solution one would need to break the replica symmetry with a larger number of steps.

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